Math 3083 Principles of Analysis — Prof. Paul Bailey

Problem Set E — Due Wednesday, December 13, 2004, 8 AM

Definition 1. Let (X, ρ) and (Y, τ) be metric spaces. Let $f : X \to Y$ and let $a \in X$.
We say that f is *continuous at a* if

$$\forall \epsilon > 0 \ \exists \delta > 0 \mid \rho(x, a) < \delta \Rightarrow \tau(f(x), f(a)) < \epsilon.$$

We say that f is *continuous* if f is continuous at a for every $a \in X$.

Definition 2. Let (X, ρ) be a metric space and let $A \subset X$.
A *cover* of A is a collection of subsets $\mathcal{C} \subset \mathcal{P}(X)$ such that $A \subset \cup\mathcal{C}$.
Let $\mathcal{C}$ be a cover of A. We say that $\mathcal{C}$ is a *finite cover* if A is a finite set. We say that $\mathcal{C}$ is an *open cover* if the elements of $\mathcal{C}$ are open sets. A *subcover* of $\mathcal{C}$ is a subset $\mathcal{D} \subset \mathcal{C}$ such that $\mathcal{D}$ is itself a cover of A.
We say that A is *compact* if every open cover of A has a finite subcover.

Definition 3. Let (X, ρ) be a metric space and let $A \subset X$.
We say that A is *disconnected* if there exist disjoint open sets U_1 and U_2 such that $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, and $A \subset U_1 \cup U_2$. We say that A is *connected* if it is not disconnected.

Fact 1. We have shown that the following statements are true regarding metric spaces.

(a) A compact set is closed and bounded.

(b) A closed subset of a compact set is compact.

(c) The continuous image of a compact set is compact.

(d) The continuous image of a connected set is connected.

(e) A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded (Heine-Borel).

(f) A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Definition 4. Let (X, ρ) be a metric space, and let A and B be nonempty subsets of X.
Define the *distance from A to B* to be

$$\rho(A, B) = \inf\{\rho(a, b) \mid a \in A, \ b \in b\}.$$

Definition 5. Let (X, ρ) be a metric space and let A and B be nonempty subsets of X.
We say that A and B are *separated* if $\rho(A, B) > 0$.

Problem 1. Let (X, ρ) be a metric space and let A and B be separated subsets of X.
Show that there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.

Problem 2. Give an example of a metric space (X, ρ) and disjoint open sets $A, B \subset X$ such that $\rho(A, B) = 0$.

Problem 3. Let (X, ρ) be a metric space and let $a \in X$. Define a function $f : X \to \mathbb{R}$ by

$$f(x) = \rho(x, a).$$

Show that f is continuous.

Problem 4. Let $A \subset \mathbb{R}$ be a closed and bounded set. Show that $\inf A \in A$ and $\sup A \in A$.

Problem 5. Let (X, ρ) be a metric space and let K be a compact subset of X.
Let $f : K \to \mathbb{R}$ be a continuous function.
Show that there exist $a, b \in K$ such that $f(a) = \min f(K)$ and $f(b) = \max f(K)$.

Problem 6. Let (X, ρ) be a metric space and let K be a compact subset of X.
Let $a \in X \smallsetminus K$. Show that $\{a\}$ and K are separated.

Problem 7. Let (X, ρ) be a metric space and let K and L be disjoint compact subsets of X.
Show that K and L are separated.